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Limit Circle Criteria and Related Properties for Nonlinear Equations*

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1. INTRODUCTION

In this paper we determine sufficient conditions for the integrability of solutions of the forced second order nonlinear differential equation

$$(a(t)x')' + q(t) f(x) = r(t) \tag{I}$$

similar in form to those known for the linear equation

$$(a(t)\, x')' + q(t)x = 0. \tag{II}$$

A special case of (I), namely, the Emden-Fowler equation

$$x'' + t^{\sigma} x^{\gamma} = 0, \tag{III}$$

$\gamma \neq 1$, will serve as our motivating prototype. In the classic paper on the subject, H. Weyl [27] classified equation (II) as being of the limit circle type if all its solutions are square integrable, i.e.,

$$\int^{\infty} x^2(u)\, du < \infty$$

for every solution $x(t)$ of (II); otherwise, the equation is said to be of the limit point type. For an excellent discussion of the limit point-limit circle problem and related matters, we refer the reader to the treatise of Dunford and Schwartz [7]. Recent papers on this problem include those by Everitt [8], Hinton [14], Knowles [16-17], Krall [18], Kwong [19], Patula *et al.* [4, 11, 21, 22],

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and Wong *et al.* [29-32]. Additional references can be found in the recently published monograph by Kauffman, Read and Zettl [15].

For the nonlinear equations (I) and (III) considerably less is known. In fact, the only references appear to be the papers of Atkinson [1], Burlak [3], Detki [6], Hallam [12], Spikes [23], Suyemoto and Waltman [24], and Wong [28]. While these other authors discuss limit point criteria for unforced equations, only Spikes [23] gives limit circle type criteria for forced equations. The exact form that integrability results for equation (I) should take is not obvious. While in the case of equation (III) with $\gamma = 2n - 1$, n a positive integer, showing that solutions belong to L^{2n} seems to be the appropriate result, for equation (I) both

$$\int^{\infty} x(u)\, f(x(u))\, du < \infty \tag{*}$$

and

$$\int^{\infty} F(x(u))\, du < \infty \qquad \text{where} \quad F(v) = \int_0^v f(u)\, du \tag{**}$$

agree with this choice and the square integrability of solutions of equation (II) as well. The integrability results presented here will insure that all solutions of (I) satisfy both (*) and (**). We will agree to say that the nonlinear equation (I) is of limit circle type if all solutions of (I) satisfy (*).

In the study of equation (II) other authors have found it convenient to introduce an appropriate transformation of the independent variable t, and we too make such a transformation on equation (I). The shape of the transformation is motivated by equation (III) and appears to be new. Moreover, it does not reduce, when $\gamma = 1$, to the transformations used by other authors for equation (II).

Section 2 contains our limit circle criteria for equation (I) and Section 3 contains results relating these ideas to the boundedness, oscillation, and convergence to zero of solutions of (I). The last section contains two limit point theorems. The first of these is related to a result obtained by Hartman and Wintner [13] for linear equations. The second limit point theorem, when combined with other results in this paper, yields necessary and sufficient conditions for all solutions of equation (I) with $f(x) = x^{2n-1}$, n a positive integer, to belong to L^{2n}.

Many of the theorems obtained in this paper are analogous to known results for linear equations. We hope that the discussion of the similarities and differences given here will encourage other researchers to further explore and develop these properties for nonlinear equations.

2. Nonlinear Limit Circle Criteria

Consider the equation

$$(a(t)x')' + q(t)\, f(x) = r(t) \tag{1}$$

where a, q, $r : [t_0, \infty) \to R$, $f : R \to R$ are continuous, a', $q' \in AC_{\text{loc}}[t_0, \infty)$, a'', $q'' \in L^2_{\text{loc}}[t_0, \infty)$, $a(t) > 0$, $q(t) > 0$, and $xf(x) \geqslant 0$ for all x. When necessary we tacitly assume that solutions of (1) are continuable; for a discussion of continuability (as well as boundedness and convergence to zero) of solutions of (1) under conditions compatible with those in this paper, we refer the reader to the papers of Graef and Spikes [9, 10]. For any continuous function h we let $h(u)_+ = \max\{h(u), 0\}$ and $h(u)_- = \max\{-h(u), 0\}$ so that $h(u) = h(u)_+ - h(u)_-$. Also, we define

$$F(v) = \int_0^v f(u)\, du.$$

We assume that there exist positive constants k and n and nonnegative constants, A, B, and K such that

$$k \geqslant 2(n+1) \tag{2}$$

$$0 \leqslant xf(x)/k - nF(x)/(n+1) \leqslant BF(x) \tag{3}$$

and

$$x^2/2 \leqslant AF(x) + K. \tag{4}$$

To simplify the notation in what follows, we let $\alpha = 1/2(n+1)$ and $\beta = (2n+1)/2(n+1)$. We make the transformation

$$s = \int_{t_0}^{t} [q^\alpha(u)/a^\beta(u)]\, du \qquad \text{and} \qquad y(s) = x(t) \tag{T_n}$$

so that equation (1) becomes

$$\ddot{y} + \alpha p(t)\dot{y} + P(t)\, f(y) = R(t) \tag{5}$$

where "$\cdot$" $= d/ds$, $p(t) = (a(t)q(t))'/a^\alpha(t)q^{\alpha+1}(t)$, $P(t) = (a(t)q(t))^{\beta-\alpha}$, and $R(t) = a^{\beta-\alpha}(t)r(t)/q^{2\alpha}(t)$. Note also that $\beta - \alpha = 2\beta - 1 = n/(n+1)$. Equation (5) can then be written as the system

$$\begin{aligned} \dot{y} &= z - p(t)y/k \\ \dot{z} &= (1/k - \alpha)\, p(t)z - P(t)\, f(y) + \{\dot{p}(t) - (1/k - \alpha)p^2(t)\}y/k + R(t) \end{aligned} \tag{6}$$

THEOREM 1. *In addition to conditions* (2)-(4) *assume that*

$$\int_{t_0}^{\infty} [(a(u)\, q(u))'_-/a(u)\, q(u)]\, du < \infty, \tag{7}$$

$$\int_{t_0}^{\infty} |\{(a(u)\, q(u))'/a^{\alpha}(u)\, q^{\alpha+1}(u)\}'$$

$$- (1/k - \alpha)[(a(u)\, q(u))']^2/a^{\alpha+1}(u)\, q^{\alpha+2}(u)|\, du < \infty, \tag{8}$$

$$\int_{t_0}^{\infty} [|\, r(u)|/(a(u)\, q(u))^{\alpha}]\, du < \infty, \tag{9}$$

and

$$\int_{t_0}^{\infty} [1/(a(u)\, q(u))^{\beta-\alpha}]\, du < \infty. \tag{10}$$

Then any solutions $x(t)$ of (1) satiesfies

$$\int_{t_0}^{\infty} F(x(u))\, du < \infty$$

and

$$\int_{t_0}^{\infty} x(u)\, f(x(u))\, du < \infty.$$

Proof. Define

$$V(y, z, s) = z^2/2 + P(t)F(y);$$

then

$$\begin{aligned} \dot{V} &= (1/k - \alpha)p(t)z^2 + \{\dot{p}(t) - (1/k - \alpha)p^2(t)\}yz/k + R(t)z \\ &\quad - P(t)p(t)\, f(y)y/k + \dot{P}(t)F(y) \\ &= (1/k - \alpha)p(t)z^2 + \{\dot{p}(t) - (1/k - \alpha)p^2(t)\}yz/k + R(t)z \\ &\quad + P(t)\{(\beta - \alpha)F(y) - f(y)y/k\}[(a(t)q(t))'/a(t)q(t)]a^{\beta}(t)/q^{\alpha}(t). \end{aligned}$$

Applying conditions (2) and (3) we have

$$\begin{aligned} \dot{V} \leqslant{} & [B - 2(1/k - \alpha)][(a(t)\, q(t))'_{-}/a(t)\, q(t)][a^{\beta}(t)/q^{\alpha}(t)]V \\ & + R(t)z + \{\dot{p}(t) - (1/k - \alpha)\, p^2(t)\}\, yz/k. \end{aligned}$$

Now $|\, yz\,| \leqslant y^2/2 + z^2/2 \leqslant z^2/2 + AF(y) + K$ by (4), so

$$\begin{aligned} \dot{V} \leqslant{} & [B - 2(1/k - \alpha)][(a(t)\, q(t))'_{-}/a(t)\, q(t)][a^{\beta}(t)/q^{\alpha}(t)]V \\ & + |\, R(t)|\, (V + 1/2) + |\, \dot{p}(t) - (1/k - \alpha)\, p^2(t)|\, (1/k)[V + K + AV/P(t)]. \end{aligned}$$

Since $\dot{p}(t) = p'(t)a^{\beta}(t)/q^{\alpha}(t)$, we have $I(t) = |\, \dot{p}(t) - (1/k - \alpha)p^2(t)\,| =$

$| p'(t) - (1/k - \alpha)p^{\hat{}}(t)q^{\alpha}(t)/a^{\beta}(t) | \, a^{\beta}(t)/q^{\alpha}(t)$. Letting $\tau(s)$ denote the inverse function of $s(t)$ we see by (8) that

$$\int_{s_0}^{s} I(\tau(\xi))\, d\xi = \int_{t_0}^{t} [I(u)\, q^{\alpha}(u)/a^{\beta}(u)]\, du$$

converges. Similarly

$$\int_{s_0}^{s} | R(\tau(\xi))|\, d\xi = \int_{t_0}^{t} [| R(u)|\, q^{\alpha}(u)/a^{\beta}(u)]\, du$$

$$= \int_{t_0}^{t} [| r(u)|/(a(u)\, q(u))^{\alpha}]\, du$$

converges by (9). Next observe that condition (7) implies that $P(t)$ is bounded from below. Hence if we integrate $\dot{V}$ from s_0 to s, use the bounds indicated above, apply Gronwall's inequality, and then transform the integrals from s back to t, we see that conditions (7)-(9) are precisely those needed to insure that $V(s)$ is bounded. Thus,

$$(a(t)q(t))^{\beta-\alpha}F(x(t)) = (a(t)q(t))^{\beta-\alpha}F(y(s)) \leqslant K_1\,,$$

for some constant $K_1 \geqslant 0$, and the conclusions of the theorem follow from conditions (10) and (3).

Remark. The strong monotonicity condition $(a(t)q(t))' \geqslant 0$ required by Spikes [23] is not needed here. Also, Theorem 1 improves a special case of a comparison result of Wong [31; Theorem 1] who needed $r \in L^2[t_0\,, \infty)$.

In order to discuss the content of Theorem 1 we first specialize the result to the well known Emden-Fowler equation

$$x'' + q(t)x^{\gamma} = 0 \tag{11}$$

where γ is an odd positive integer and q is as before. Actually, the corollary stated below results from applying the transformation (T_n) and the technique of proof used above directly to equation (11) (the n in (T_n) is chosen so that $\gamma = 2n - 1$). Conditions (2) and (3) are satisfied if we take $k = 2(n + 1)$ and $B = 0$; clearly, (4) holds (for example, let $A = n$ and $K = 1$). Since $k = 2(n + 1)$ we see from the proof of Theorem 1 that condition (7) is not needed. However, unless we ask that q be bounded from below, we must require the additional integral condition (13) below. Hence we have:

COROLLARY 2. *Assume that*

$$\int_{t_0}^{\infty} | q''(u)/q^{\theta}(u) - \theta[q'(u)]^2/q^{\phi}(u)|\, du < \infty$$

and

$$\int_{t_0}^{\infty} \{| q''(u)/q^{\theta}(u) - \theta[q'(u)]^2/q^{\phi}(u)|/q^{n/(n+1)}(u)\}\, du < \infty \tag{13}$$

where $\theta = (2n + 3)/2(n + 1)$ *and* $\phi = (4n + 5)/2(n + 1)$. *if*

$$\int_{t_0}^{\infty} [1/q^{n/(n+1)}(u)]\, du < \infty \tag{14}$$

then every solution $x(t)$ *of* (11) *satiesfies* $\int_{t_0}^{\infty} x^{2n}(u)\, du < \infty$.

To see that condition (10) is sharp, consider the special case of (11) when $q(t) = t^{\sigma}$, namely,

$$x'' + t^{\sigma} x^{2n-1} = 0. \tag{15}$$

Now (10) implies that $\sigma n/(n + 1) > 1$, or

$$\sigma > 1 + 1/n. \tag{16}$$

By asymptotic integrations it is known that all solutions of (15) belong to $L^{2n}[t_0, \infty)$ only if (16) holds (see Atkinson [1; p. 311] or Bellman [2; p. 163]). Since (12) and (13) are obviously satisfied in this case, we have that (16) is both necessary and sufficient for all solutions of (15) to belong to $L^{2n}[t_0, \infty)$.

If $n = 1$ so that equation (11) is linear,

$$x'' + q(t)x = 0, \tag{17}$$

then condition (12) becomes

$$\int_{t_0}^{\infty} | q''(u)/q^{5/4}(u) - (5/4)[q'(u)]^2/q^{9/4}(u)|\, du < \infty \tag{18}$$

and (13) becomes

$$\int_{t_0}^{\infty} | q''(u)/q^{7/4}(u) - (5/4)[q'(u)]^2/q^{11/4}(u)|\, du < \infty. \tag{19}$$

While for large $q(t)$ condition (18) is not as good as the criteria of Dunford and Schwartz [7; p. 1414],

$$\int_{t_0}^{\infty} | q''(u)/q^{3/2}(u) - (5/4)[q'(u)]^2/q^{5/2}(u)|\, du < \infty \tag{20}$$

(see also, for example, Burton *et al.* [4]), condition (19) is better. (We say "for large $q(t)$" because it is known [13, 20] that equation (17) is of limit point type if $q(t)$ is bounded from above.) Since, however, inequality (4) is not needed when the equation is linear, we could reconstruct V from our expression for $\dot{V}$ by using the inequality

$$I(t) \mid yz \mid \leqslant [I(t)/q^{1/4}(t)][q^{1/2}(t)y^2/2 + z^2/2] = [I(t)/q^{1/4}(t)]V.$$

Thus, instead of (18) and (19) we would have needed precisely the single condition (20) of Dunford and Schwartz. This method of reconstructing V from $\dot{V}$ does not yield as good a result for nonlinear equations as does the technique used in the proof of Theorem 1. For example, for equation (1) condition (8) would have to be replaced by

$$\int_{t_0}^{\infty} |\{(a(u)\, q(u))'/a^{\alpha}(u)\, q^{\alpha+1}(u)\}' \\ - (1/k - \alpha)[(a(u)\, q(u))']^2/a^{\alpha+1}(u)\, q^{\alpha+2}(u)|\; (a(u)\, q(u))^{(\beta-\alpha)/2}\, du < \infty,$$

and for equation (11) condition (12) would have to be replaced by

$$\int_{t_0}^{\infty} \mid q''(u)/q^{\lambda}(u) - \theta[q'(u)]^2\, q^{\psi}(u)|\; du < \infty$$

where $\lambda = (n + 3)/2(n + 1)$ and $\psi = (3n + 5)/2(n + 1)$, which again are not as good as (8) and (12) when $a(t)q(t)$ and $q(t)$ respectively are large. Finally we note that in the case of the linear equation (17), condition (14) reduces to

$$\int_{t_0}^{\infty} [1/q^{1/2}(u)]\; du < \infty$$

which, under the covering assumption (20), is known to be necessary and sufficient for equation (17) to be of limit circle type. In this regard, it would be interesting to know whether or not condition (10) is necessary for the nonlinear equation (1) to be limit circle.

Remark. Limit circle criteria under conditions similar to (20) can be found in the papers of Burton and Patula [4], Everitt [8], Hinton [14], Knowles [16, 17] and Wong [29], and the monographs of Coppel [5] and Titchmarsh [25]. For the linear equation

$$(a(t)x')' + q(t)x = 0$$

Theorem 1, and in particular condition (8), is more easily compared to the results in [8], [14], or [29] than those in Dunford and Schwartz [7] since, in this case, we conveniently chose $1/k = \alpha$.

3. Relationship to Other Properties

In this section we discuss some relationships between the nonlinear limit circle property and the boundedness, oscillation, and convergence to zero of solutions of (1). For linear equations such relationships have been discussed by other authors. For example, in a recent paper Burton and Patula [4] assumed that equation (17) was limit circle and then asked what additional conditions are needed to insure that all solutions are bounded. (Wong [30; p. 284] conjectured that if equation (17) is limit circle, then all solutions must be bounded. This has been shown to be false [19].) This same problem was discussed by Krall [18] for the more general equation (1) with $f(x) = x$. Theorems relating oscillation and the limit circle property can be found in the papers of Grimmer and Patula [11], Patula and Waltman [21], and Patula and Wong [22]. Results showing when limit circle equations have all solutions converging to zero are given by Burton and Patula [4]. We will begin with some consequences of the integrability criteria given in Theorem 1.

Theorem 3. *Under the hypotheses of Theorem* 1 *all solutions of equation* (1) *are bounded. If in addition $F(x) > 0$ if $x \neq 0$ and $a(t)q(t) \to \infty$ as $t \to \infty$, then all solutions of* (1) *converge to zero as $t \to \infty$.*

Proof. From the proof of Theorem 1 we have

$$(a(t)q(t))^{\beta-\alpha}F(x(t)) \leqslant K_1$$

Condition (7) implies that $a(t)q(t)$ is bounded below away from zero, so $F(x(t))$ is bounded. The boundedness of $x(t)$ now follows from condition (4). If $a(t)q(t) \to \infty$ as $t \to \infty$, then the above inequality shows that $F(x(t)) \to 0$ as $t \to \infty$. Since $F(x) > 0$ for $x \neq 0$, we have that $x(t) \to 0$ as $t \to \infty$.

Remark. From the first part of Theorem 3 we have that all solutions of (1) belong to $L^\infty[t_0, \infty)$. The remark following the proof of Theorem 1 now applies to Theorem 2 in [31] (see also Theorem B in [30]).

The following theorem is similar to a result of Burton and Patula [4].

Theorem 4. *Suppose that there exist constants $M > 0$, $N \geqslant 0$, $m > 0$ and $b \geqslant 0$ such that*

$$x^2 \leqslant Mxf(x) + N, \tag{21}$$

$$|a^{1/2}(t)\, q'(t)/q^{3/2}(t)| \leqslant m, \tag{22}$$

$$|r(t)/q(t)| \leqslant b, \tag{23}$$

$$\int_{t_0}^{\infty} |r(u)/q(u)|\, du < \infty, \tag{24}$$

and

$$\int_{t_0}^{\infty} \{a(u)[q'(u)]^2/q^3(u)\}\, du < \infty. \tag{25}$$

If $x(t)$ *is a limit circle solution of* (1), *i.e.,*

$$\int_{t_0}^{\infty} x(u)\, f(x(u))\, du < \infty, \tag{26}$$

then

$$\int_{t_0}^{\infty} \{a(u)[x'(u)]^2/q(u)\}\, du < \infty. \tag{27}$$

Proof. Multiplying equation (1) by $x(t)/q(t)$, noticing that $(a(t)x')'x = (a(t)x'x)' - a(t)[x']^2$, and integrating by parts we have

$$\begin{aligned} &a(t)\, x'(t)\, x(t)/q(t) - a(t_1)\, x'(t_1)\, x(t_1)/q(t_1) \\ &\quad + \int_{t_1}^{t} [a(u)\, x'(u)\, x(u)\, q'(u)/q^2(u)]\, du + \int_{t_1}^{t} x(u)\, f(x(u))\, du \\ &\quad - \int_{t_1}^{t} \{a(u)[x'(u)]^2/q(u)\}\, du = \int_{t_1}^{t} [x(u)\, r(u)/q(u)]\, du \end{aligned} \tag{28}$$

for any $t_1 \geqslant t_0$. By the Schwartz inequality

$$\begin{aligned} &\left| \int_{t_1}^{t} [a(u)\, x'(u)\, x(u)\, q'(u)/q^2(u)]\, du \right| \\ &\qquad \leqslant \left[\int_{t_1}^{t} \{a(u)[x'(u)]^2/q(u)\}\, du\right]^{1/2} \\ &\qquad\qquad \times \left[\int_{t_1}^{t} \{a(u)\, x^2(u)[q'(u)]^2/q^3(u)\}\, du\right]^{1/2}. \end{aligned}$$

Now from (21) and (22) we have $a(t)x^2(t)[q'(t)]^2/q^3(t) \leqslant m^2Mx(t)\, f(x(t)) + Na(t)[q'(t)]^2/q^3(t)$, so integrating and applying (25) and (26) we obtain

$$\int_{t_1}^{\infty} \{a(u)\, x^2(u)[q'(u)]^2/q^3(u)\}\, du \leqslant K_1 < \infty.$$

Since

$$\begin{aligned} |\, x(t)\,|\, r(t)/q(t) &\leqslant x^2(t)r(t)/2q(t) + r(t)/2q(t) \\ &\leqslant Mbx(t)\, f(x(t))/2 + r(t)(N+1)/2q(t), \end{aligned}$$

we see that the integral on the right hand side of (28) converges by virtue of (24) and (26). If $x(t)$ is not eventually monotonic, let $\{t_j\} \to \infty$ be an increasing sequence of zeros of $x'(t)$. Then from (28) we have

$$K_1 H^{1/2}(t_j) + K_2 \geqslant H(t_j)$$

where

$$H(t) = \int_{t_1}^{t} \{a(u)[x'(u)]^2/q(u)\}\, du.$$

It follows that $H(t_j) \leqslant K_3 < \infty$ for all j and so (27) holds.

If $x(t)$ is eventually monotonic, then $x(t)x'(t) \leqslant 0$ for $t \geqslant t_1$ for sufficiently large $t_1 \geqslant t_0$ since otherwise condition (26) would be violated. Using this fact in (28) we can repeat the type of argument used above to again obtain that (27) holds.

Theorem 4 was proved in a slightly more general form by Burton and Patula [4; Lemma 2] for equation (17). Clearly, in the linear case condition (21) is not needed, and, as we see from the above proof, this renders condition (25) unnecessary as well. Condition (22) has been used by many authors when discussing limit point and limit circle criteria for linear equations.

Under rather mild restrictions, Patula and Waltman [21; Theorem 1] proved that the linear equation

$$(a(t)x')' + q(t)x = 0$$

is oscillatory if it is limit circle. This is not true in general for forced equations. For example, Theorem 1 guarantees that all solutions of

$$(tx')' + t^2 x = 9/t^4 + 1/t,\ t \geqslant 1$$

belong to $L^2[1, \infty)$, but $x(t) = 1/t^3$ is a nonoscillatory solution of this equation. (It follows from Theorem 2 in [11] that his is the only nonoscillatory solution.) However, we are able to prove the following result for forced equations.

THEOREM 5. *Assume that $f(x)$ is bounded away from zero if x is bounded away from zero,*

$$\int_{t_0}^{\infty} \left(\int_{t_0}^{u} [1/a(v)]\, dv\right) |r(u)|\, du < \infty, \tag{29}$$

and equation (1) is limit circle in the sense that every solution of (1) satisfies (26). Then each solution of (1) either oscillates or converges monotonically to zero as $t \to \infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1), say $x(t) > 0$ for $t \geqslant t_1 \geqslant t_0$.

Clearly $\liminf_{t\to\infty} x(t) = 0$ for otherwise condition (26) would be violated. If $x(t)$ is eventually monotonic, we are done, so we assume that $x(t)$ is not eventually monotonic.

If $x(t)$ does not converge to zero as $t \to \infty$, then there exists $K_1 > 0$ such that for any $t_2 > t_1$ there exists $t_3 > t_2$ with $x(t_3) \geqslant K_1$. Choose $t_2 > t_1$ so that

$$\int_{t_2}^{\infty} \left(\int_{t_2}^{u} [1/a(v)]\, dv\right) |\, r(u)|\, du < K_1/2$$

and choose $t_3 > t_2$ such that $x'(t_3) = 0$ and $x(t_3) \geqslant K_1$. Integrating equation (1) we have

$$x'(t) = [1/a(t)] \int_{t_3}^{t} [r(u) - q(u) f(x(u))]\, du. \tag{30}$$

Another integration yields

$$x(t) = x(t_3) + \left(\int_{t_3}^{t} [1/a(u)]\, du\right) \int_{t_3}^{t} [r(u) - q(u) f(x(u))]\, du$$

$$- \int_{t_3}^{t} \left(\int_{t_3}^{u} [1/a(v)]\, dv\right) [r(u) - q(u) f(x(u))]\, du.$$

If $t_4 > t_3$ is any zero of $x'(t)$, then (30) shows that the first integral above vanishes, and so we have

$$x(t_4) \geqslant x(t_3) - \int_{t_3}^{t_4} \left(\int_{t_3}^{u} [1/a(v)]\, dv\right) |\, r(u)|\, du \geqslant K_1/2.$$

That is, $x(t)$ is bounded below by $K_1/2$ at every zero of $x'(t)$ for $t \geqslant t_3$ which contradicts $\liminf_{t\to\infty} x(t) = 0$. The proof in case $x(t) < 0$ for $t \geqslant t_1$ is similar.

If $x(t)$ is a solution of (1) with arbitrarily large zeros but is ultimately nonnegative or nonpositive (Z-type solutions as defined in [9] or [10]), the argument used in the proof of Theorem 5 shows that such solution also converge to zero. The proof of Theorem 5 is similar to the proof of Theorem 8 in [9]. Considerably fewer explicit hypotheses are needed here due to the strength of the limit circle assumption on equation (1). Condition (29) is not necessary for nonoscillatory solutions to converge to zero. For example, the forcing term in the equation preceding Theorem 5 does not satisfy (29). Other criteria for the convergence to zero of oscillatory and nonoscillatory solutions can be found in [9] and [10].

If $r(t) \equiv 0$, then it is possible to obtain the stronger result that all solutions must in fact oscillate. In this regard we have the following theorem.

THEOREM 6. *If in addition to the hypotheses of Theorem 5 we assume that* $r(t) \equiv 0$ *and*

$$\int_{t_0}^{\infty} [1/a(u)]\, du = \infty,$$

then all solutions of (1) *oscillate.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1), say $x(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. The proof in case $x(t) < 0$ for $t \geqslant t_1$ is similar and will be omitted. Assume that there exists $t_2 \geqslant t_1$ such that $x'(t_2) < 0$. Then from equation (1) we have $(a(t)x'(t))' \leqslant 0$, and integrating twice we obtain

$$x(t) \leqslant x(t_2) + \int_{t_2}^{t} [a(t_2)\, x'(t_2)/a(u)]\, du \to -\infty$$

as $t \to \infty$. This contradicts the fact that $x(t) > 0$ for $t \geqslant t_1$. Thus $x'(t) \geqslant 0$ for $t \geqslant t_1$ which leads immediately to a contradiction of (26).

Remark. Theorem 6 extends Theorem 2(b) of Wong and Zettl [32].

It is possible to obtain a result of this type for forced equations without requiring $r(t)$ to be small as in the case of condition (29). Instead we ask that $r(t)$ oscillates in a certain fashion.

THEOREM 7. *Suppose that the hypotheses of Theorem 5 hold except possibly for condition* (29). *If for every* $c > 0$ *and all large* $\epsilon \geqslant t_0$ *we have*

$$\liminf_{t\to\infty} \left\{\int_{\epsilon}^{t} [1/a(u)] \left[\int_{\epsilon}^{u} r(v)\, dv - c\right] du\right\} = -\infty \tag{31}$$

and

$$\limsup_{t\to\infty} \left\{\int_{\epsilon}^{t} [1/a(u)] \left[\int_{\epsilon}^{u} r(v)\, dv + c\right] du\right\} = +\infty,$$

then all solutions of (1) *are oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1) with $x(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. Choose $t_2 \geqslant t_1$ so that (31) holds. If $x'(t) \geqslant 0$ for $t \geqslant t_2$, then we would obtain an immediate contradiction to (26). Hence there exists $t_3 \geqslant t_2$ such that $x'(t_3) < 0$. Letting $a(t_3)x'(t_3) = -c < 0$ and integrating equation (1) twice we have

$$x(t) \leqslant x(t_3) + \int_{t_3}^{t} [1/a(u)] \left[\int_{t_3}^{u} r(v)\, dv - c\right] du.$$

Condition (31) implies that $x(t_4) < 0$ for some $t_4 > t_3$, and this contradicts $x(t) > 0$ for $t \geqslant t_1$. A similar proof holds if $x(t) < 0$ for $t \geqslant t_1$.

4. Limit Point Criteria

Limit point criteria for second order linear equations can be found in the works of many authors. Among the best known results of this type are those of Hartman and Wintner [13] and Levinson [20]. Titchmarsh [25, 26] discusses an important relationship between the existence of a limit point solution and the solution of certain boundary value problems. The importance of obtaining that equation (1) is of limit point type is pointed out in [13]. As a consequence of one of the limit point theorems given in this section, we are able to obtain necessary and sufficient conditions for a special case of equation (1) to be limit circle.

To simplify the notation used in the following theorem, we define the functions $h, H : [t_0, \infty) \to R$ by

$$h(t) = (a(t)\, q(t))'_-/a(t)\, q(t) + |\, r(t)|$$

and

$$H(t) = (a(t)\, q(t))'_+/a(t)\, q(t) + |\, r(t)|/(a(t)\, q(t))^{1/2}.$$

THEOREM 8. *Suppose that conditions* (21)-(25) *hold, there exists* $M_1 > 0$ *such that*

$$F(x) \leqslant M_1 x f(x), \tag{32}$$

and either

$$\text{(i)} \quad \int_{t_0}^{\infty} [1/a(u)\, q(u)] \exp\left(-\int_{t_0}^{u} h(v)\, dv\right) du = \infty,$$

and

$$\int_{t_0}^{\infty} |\, r(u)| \exp \int_{t_0}^{u} h(v)\, dv\, du < \infty \tag{33}$$

or,

$$\text{(ii)} \quad \int_{t_0}^{\infty} \exp\left(-\int_{t_0}^{u} H(v)\, dv\right) du = \infty$$

and

$$\int_{t_0}^{\infty} [|\, r(u)|/(a(u)\, q(u))^{1/2}] \exp \int_{t_0}^{u} H(v)\, dv\, du < \infty.$$

Then equation (1) *is of limit point type, i.e., there is a solution of* (1) *wich does not satisfy* (26).

Proof. We will write equation (1) as the system

$$\begin{aligned} x' &= y \\ y' &= (-\,a'(t)y - q(t)\,f(x) + r(t))/a(t). \end{aligned} \tag{34}$$

If (i) holds, define $V(x, y, t) = a^2(t)y^2(t)/2 + a(t)q(t)F(x)$. Then

$$V' = a(t)\,r(t)y + (a(t)\,q(t))'\,F(x) \geqslant a(t)\,r(t)y - [(a(t)\,q(t))'_-/a(t)\,q(t)]V.$$

Now

$$|\,a(t)\,r(t)y\,| \leqslant |\,r(t)|\,[a^2(t)\,y^2/2 + 1/2]$$

so

$$V' \geqslant -[(a(t)\,q(t))'_-/a(t)\,q(t) + |\,r(t)|]V - |\,r(t)|/2. \tag{35}$$

From condition (33) we have

$$(1/2)\int_{t_0}^{\infty} |\,r(u)|\,\exp\int_{t_0}^{u} h(v)\,dv\,du \leqslant K_1 < \infty,$$

so let $(x(t), y(t))$ be a solution of (34) such that $(x(t_0), y(t_0)) = (x_0\,, y_0)$ and $V(x_0\,, y_0\,, t_0) = V(t_0) > K_1 + 1$. Then from (35) we have

$$\left(V(t)\exp\int_{t_0}^{t} h(u)\,du\right)' \geqslant -(|\,r(t)|/2)\exp\int_{t_0}^{t} h(u)\,du.$$

Integrating we obtain

$$V(t)\exp\int_{t_0}^{t} h(u)\,du \geqslant V(t_0) - K_1 > 1.$$

Hence,

$$\int_{t_0}^{t} [V(u)/a(u)\,q(u)]\,du \geqslant \int_{t_0}^{t} [1/a(u)\,q(u)]\exp\left(-\int_{t_0}^{u} h(v)\,dv\right)du \to \infty$$

as $t \to \infty$. In view of Theorem 4 this shows that $x(t)$ cannot be a limit circle solution of (1).

If (ii) holds, define $V_1(x, y, t) = a(t)y^2/2q(t) + F(x)$. Then $V_1' \geqslant r(t)y/q(t) - [(a(t)q(t))_+/a(t)q(t)]V_1$. Since $|\,r(t)y/q(t)\,|\,[|\,r(t)\,|/(a(t)q(t))^{1/2}](V_1 + 1/2)$, we have

$$V_1' + H(t)V_1 \geqslant -\,|\,r(t)\,|\,/2(a(t)q(t))^{1/2}.$$

The remainder of the proof is similar to the proof when (i) holds and is omitted.

Remark. A result similar to Theorem 8 was proved by Wong and Zettl

[32; Theorem 1] for equation (1) with $f(x) = x$ and $r(t) \equiv 0$. Their conditions imply (i) and (ii) of Theorem 8, but, on the other hand, they do not require condition (22).

The following corollary is an immediate consequence of the proof of the above theorem.

COROLLARY 9. *If Theorem 8 we have $r(t) \equiv 0$, then no nontrivial solution of* (1) *satisfies* (26).

Remark. When $f(x) = x$, Corollary 9 generalizes a result of Hartman and Wintner [13; p. 303] who required that $a(t) \equiv 1$ and that $q(t)$ be monotonic. Note also that condition (32) is not an unreasonable assumption. It is satisfied, for example, if $f'(x) \geqslant 0$ and $M_1 \geqslant 1$.

In the proof of our next limit point theorem it will be convenient to have the following lemma at our disposal.

LEMMA 10. *In addition to condition* (21) *assume that there exists $m_1 > 0$ such that*

$$|(a(t)q(t))'/a^{1/2}(t)q^{3/2}(t)| \leqslant m_1 \tag{36}$$

and

$$\int_{t_0}^{\infty} \{[(a(u)\, q(u))']^2/a(u)\, q^3(u)\}\, du < \infty. \tag{37}$$

If $x(t)$ is a limit circle solution of (1) *(i.e.,* (26) *holds), then*

$$\int_{t_0}^{\infty} \{[(a(u)\, q(u))']^2\, x^2(u)/a(u)\, q^3(u)\}\, du < \infty.$$

Proof. We have

$$\int_{t_0}^{\infty} \{[(a(u)\, q(u))']^2\, x^2(u)/a(u)\, q^3(u)\}\, du$$

$$\leqslant m_1{}^2 M \int_{t_0}^{\infty} x(u)\, f(x(u))\, du + \int_{t_0}^{\infty} \{[(a(u)\, q(u))']^2/a(u)\, q^3(u)\}\, du < \infty$$

by (26) and (37).

THEOREM 11. *Suppose that there exist constants $k_1 > 0$, $n > 0$, and $B_1 \geqslant 0$ such that*

$$k_1 \leqslant 2(n+1) \tag{38}$$

and

$$0 \leqslant (\beta - \alpha)F(x) - xf(x)/k_1 \leqslant B_1 F(x). \tag{39}$$

In addition assume that condition (4), (7)-(9), (22)-(25), (32), (36), *and* (37) *hold with the k in condition* (8) *replaced by* k_1. *If*

$$\int_{t_0}^{\infty} [1/(a(u)\, q(u))^{\beta-\alpha}]\, du = \infty, \tag{40}$$

then equation (1) *is of limit point type.*

Proof. Apply the transformation (T_n) to equation (1) to obtain equation (5), and then write (5) in the form of system (6) with k now replaced by k_1. As in the proof of Theorem 1 we define

$$V(y, z, s) = z^2/2 + (a(t)q(t))^{\beta-\alpha}F(y)$$

and differentiate to obtain

$$\begin{aligned}
\dot V \geqslant &- (1/k_1 - \alpha)(a(t)\, q(t))'_-\, z^2/a^\alpha(t)\, q^{\alpha+1}(t) - |\, R(t)|\, [V + 1/2]\\
&- |\, \dot p(t) - (1/k_1 - \alpha)\, p^2(t)|\, [V + K + AV/P(t)]/k_1\\
&- P(t)\{(\beta - \alpha)\, F(y) - f(y)y/k_1\}[(a(t)\, q(t))'_-/a(t)\, q(t)]\, a^\beta(t)/q^\alpha(t)\\
\geqslant &- [B_1 + 2(1/k_1 - \alpha)][(a(t)\, q(t))'_-/a(t)\, q(t)][a^\beta(t)/q^\alpha(t)]V\\
&- |\, \dot p(t) - (1/k_1 - \alpha)\, p^2(t)|\, [V + K + AV/P(t)]/k_1\\
&- |\, R(t)|\, [V + 1/2].
\end{aligned}$$

Now define the functions $G, g : [t_0, \infty) \to R$ by

$$\begin{aligned}
G(t) = \{&[B_1 + 2(1/k_1 - \alpha)](a(t)\, q(t))'_-/a(t)\, q(t) + |\, r(t)|/(a(t)\, q(t))^\alpha\\
&+ |\, p'(t) - (1/k_1 - \alpha)\, p^2(t)\, q^\alpha(t)/a^\beta(t)|\, [1 + A/P(t)]/k_1\}\, a^\beta(t)/q^\alpha(t)
\end{aligned}$$

and

$$\begin{aligned}
g(t) = \{&(K/k_1)\, |\, p'(t) - (1/k_1 - \alpha)p^2(t)q^\alpha(t)/a^\beta(t)\, |\\
&+ |\, r(t)\, |\, /2(a(t)q(t))^\alpha\}a^\beta(t)/q^\alpha(t).
\end{aligned}$$

We can then write

$$\dot V + G(t)V \geqslant -g(t)$$

and so

$$\left(V \exp \int_{s_0}^{s} G(\tau(\xi))\, d\xi\right)^{\cdot} \geqslant -g(t) \exp \int_{s_0}^{s} G(\tau(\xi))\, d\xi \geqslant -K_1 g(t) \tag{41}$$

since conditions (7)-(9) guarantee that

$$\exp \int_{s_0}^{\infty} G(\tau(\xi))\, d\xi \leqslant K_1 < \infty$$

for some constant $K_1 > 0$. Conditions (8) and (9) imply that

$$K_1 \int_{s_0}^{\infty} g(\tau(\xi))\, d\xi \leqslant K_2 < \infty$$

for some $K_2 > 0$, so let $x(t)$ be any solution of (1) such that $V(y(s_0), z(s_0), s_0) > K_2 + 1$. Integrating (41) we have

$$V(s) \exp \int_{s_0}^{s} G(\tau(\xi))\, d\xi \geqslant V(s_0) - K_2 > 1$$

and thus

$$V(s) \geqslant 1/K_1$$

for $s \geqslant s_0$. Dividing both members of this last inequality by $(a(t)q(t))^{\beta-\alpha}$ and rewriting the left hand side in terms of t we have

$$\begin{aligned} a(t)[x'(t)]^2/2q(t) &+ (a(t)q(t))'x(t)x'(t)/kq^2(t) \\ &+ [(a(t)q(t))']^2x^2(t)/2k^2a(t)q^3(t) \\ &+ F(x(t)) \geqslant 1/K_1(a(t)q(t))^{\beta-\alpha}. \end{aligned} \tag{42}$$

If $x(t)$ was a limit circle solution of (1), then since conditions (4) and (32) imply (21) we have

$$\int_{t_0}^{\infty} \{a(u)[x'(u)]^2/q(u)\}\, du < \infty$$

by Theorem 4. Also,

$$\int_{t_0}^{\infty} \{[(a(u)\, q(u))']^2\, x^2(u)/a(u)\, q^3(u)\}\, du < \infty$$

by Lemma 10,

$$\begin{aligned} &\left| \int_{t_0}^{\infty} \{(a(u)\, q(u))'\, x(u)\, x'(u)/q^2(u)\}\, du \right| \\ &\qquad \leqslant \left[\int_{t_0}^{\infty} \{[(a(u)\, q(u))']^2\, x^2(u)/a(u)\, q^3(u)\}\, du \right]^{1/2} \\ &\qquad\quad \cdot \left[\int_{t_0}^{\infty} \{a(u)[x'(u)]^2/q(u)\}\, du \right]^{1/2} < \infty \end{aligned}$$

by Lemma 10 and Theorem 4, and

$$\int_{t_0}^{\infty} F(x(u))\, du < \infty$$

by condition (32) and the supposition that $x(t)$ was a limit circle solution. Thus by integrating (42) we have a contradiction.

By combining Theorem 11 with Theorems 1 and 4 we can obtain necessary and sufficient conditions for the equation

$$(a(t)x')' + q(t)x^{2n-1} = r(t), \tag{43}$$

n a positive integer, to be of limit circle type, i.e., all solutions belong to $L^{2n}[t_0, \infty)$. Note first that conditions (2), (3), (38), and (39) are satisfied with $k = k_1 = 2(n + 1)$ and $B = B_1 = 0$. In addition, conditions (4) and (32) are automatically satisfied.

THEOREM 12. *Assume that conditions* (7), (9), (22)-(25), (36), *and* (37) *hold, and*

$$\int_{t_0}^{\infty} |\{(a(u)\, q(u))'/a^{\alpha}(u)\, q^{\alpha+1}(u)\}'\,|\, du < \infty.$$

Then equation (43) *is of limit circle type if and only if*

$$\int_{t_0}^{\infty} [1/(a(u)\, q(u))^{n/(n+1)}]\, du < \infty. \tag{44}$$

When we specialize this theorem to the equation

$$x'' + q(t)x^{2n-1} = 0. \tag{45}$$

we obtain the following result.

COROLLARY 13. *If*

$$\int_{t_0}^{\infty} [q'(u)_-/q(u)]\, du < \infty, \tag{46}$$

$$|\, q'(t)/q^{3/2}(t)| \leqslant m_2\,,$$

$$\int_{t_0}^{\infty} \{[q'(u)]^2/q^3(u)\}\, du < \infty,$$

and condition (12) *holds, then equation* (45) *is of limit circle type if and only if*

$$\int_{t_0}^{\infty} [1/q^{n/(n+1)}(u)]\, du < \infty. \tag{47}$$

Remark. Much of the discussion in Section 2 applies to Theorem 11, Theorem 12, and Corollary 13 as well. For example, condition (46) can be replaced by condition (13).

If $q(t) = t^{\sigma}$ in equation (45), then

$$\int_{t_0}^{\infty} [1/q^{n/(n+1)}(u)]\, du = \infty$$

implies that $\sigma n/(n + 1) \leqslant 1$, or,

$$\sigma \leqslant 1 + 1/n$$

(compare this with (16)). This is in complete agreement with Atkinson's [1] results, and leads us to believe that conditions (40), (44), and (47) in the above theorems are sharp.

References

1. F. V. Atkinson, Nonlinear extensions of limit-point criteria, *Math. Z.* **130** (1973), 297–312.
2. R. Bellman, "Stability Theory of Differential Equations," McGraw–Hill, New York, 1953.
3. J. Burlak, On the non-existence of L_2-solutions of nonlinear differential equations, *Proc. Edinburgh Math. Soc.* 14 (1965), 257–268.
4. T. A. Burton and W. T. Patula, Limit circle results for second order equations, *Monatsh. Math.* 81 (1976), 185–194.
5. W. A. Coppel, "Stability and Asymptotic Behavior of Differential Equations," Heath, Boston, 1965.
6. J. Detki, The solvability of a certain second order nonlinear ordinary differential equation in $L^p(0, \infty)$, *Math. Balk.* 4 (1974), 115–119.
7. N. Dunford and J. T. Schwartz, "Linear Operators," Part II, "Spectral Theory," Interscience, New York, 1963.
8. W. N. Everitt, On the limit-circle classification of second-order differential expressions, *Quart. J. Math. Oxford Ser.* (2) **23** (1972), 193–196.
9. J. R. Graef and P. W. Spikes, Asymptotic behavior of solutions of a second order nonlinear differential equation, *J. Differential Equations* **17** (1975), 461–476.
10. J. R. Graef and P. W. Spikes, Asymptotic properties of solutions of a second order nonlinear differential equation, *Publ. Math. Debrecen* 24 (1977), 39–51.
11. R. C. Grimmer and W. T. Patula, Nonoscillatory solutions of forced second-order linear equations, *J. Math. Anal. Appl.* **56** (1976), 452–459.
12. T. G. Hallam, On the nonexistence of L^p solutions of certain nonlinear differential equations, *Glasgow Math. J.* 8 (1967), 133–138.

13. P. HARTMAN AND A. WINTNER, Criteria of non-degeneracy for the wave equation, *Amer. J. Math.* **70** (1948), 295–308.
14. D. HINTON, Limit point–limit circle criteria for $(py')' + qy = \lambda ky$, *in* "Ordinary and Partial Differential Equations," pp. 173–183, Lecture Notes in Mathematics No. 415, Springer-Verlag, New York, 1974.
15. R. M. KAUFFMAN, T. T. Read, AND A. ZETTL, "The Deficiency Index Problem for Powers of Ordinary Differential Expressions," Lecture Notes in Mathematics No. 621, Springer–Verlag, New York, 1977.
16. I. KNOWLES, On a limit-circle criterion for second-order differential operators, *Quart. J. Math. Oxford Ser.* (2) **24** (1973), 451–455.
17. I. KNOWLES, On second-order differential operators of limit circle type, *in* "Ordinary and Partial Differential Equations," pp. 184–187, Lecture Notes in Mathematics No. 415, Springer–Verlag, New York, 1974.
18. A. M. KRALL, On the solutions of $(ry')' + qy = f$, *Monatsh. Math.* **80** (1975), 115–118.
19. M. K. KWONG, On boundedness of solutions of second order differential equations in the limit circle case, *Proc. Amer. Math. Soc.* **52** (1975), 242–246.
20. N. LEVINSON, Criteria for the limit-point case for second order linear differential operators, *Časopis Pěst. Mat.* **74** (1949), 17–20.
21. W. T. PATULA AND P. WALTMAN, Limit point classification of second order linear differential equations, *J. London Math. Soc.* (2) **8** (1974), 209–216.
22. W. T. PATULA AND J. S. W. WONG, An L^p-analogue of the Weyl alternative, *Math. Ann.* **197** (1972), 9–28.
23. P. W. SPIKES, On the integrability of solutions of perturbed nonlinear differential equations, *Proc. Roy. Soc. Edinburgh Sect. A* **77** (1977), 309–318.
24. L. SUYEMOTO AND P. WALTMAN, Extension of a theorem of A. Winter, *Proc. Amer. Math. Soc.* **14** (1963), 970–971.
25. E. C. TITCHMARSH, "Eigenfunction Expansions Associated with Second-Order Differential Equations," Oxford Univ. Press, Oxford, 1946.
26. E. C. TITCHMARSH, On the uniqueness of the Green's function associated with a second-order differential equation, *Canad. J. Math.* **1** (1949), 191–198.
27. H. WEYL, Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörige Entwicklung willkürlicher Funktionen, *Math. Ann.* **68** (1910), 220–269.
28. J. S. W. WONG, Remark on a theorem of A. Wintner, *Enseignement Math.* (2) **13** (1967), 103–106.
29. J. S. W. WONG, Remarks on the limit-circle classification of second-order differential operators, *Quart. J. Math. Oxford* (2) **24** (1973), 423–425.
30. J. S. W. WONG, Square integrable solutions of L^p perturbations of second order linear differential equations, *in* "Ordinary and Partial Differential Equations," pp. 282–292, Lecture Notes in Mathematics No. 415, Springer–Verlag, New York, 1974.
31. J. S. W. WONG, Square integrable solutions of perturbed linear differential equations, *Proc. Roy. Soc. Edinburgh Sect. A* **73** (1974–1975), 251–254.
32. J. S. W. WONG AND A. ZETTL, On the limit point classification of second order differential equations, *Math. Z.* **132** (1973), 297–304.